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MATH+

A fully computer-assisted approach in Extremal Combinatorics

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Erdős' Ramsey Multiplicity Conjecture

The field of Ramsey theory is concerned with **finding order in any large enough system**. The Ramsey-number $R(s,t)$ is the smallest integer n such that any graph on n vertices contains a complete subgraph of size s or an independent set of size t.

Ramsey's Theorem (1930)

$R(s,t)$ exists for any integers s and t.

Determining $R(s,t)$, either for specific values of s,t or asymptotically as either or both tend to infinity, is a problem with a long history. A related problem asks for the **minimum density of copies of the complete subgraph K_s and independent sets of size t in any n-vertex graphs**. While for $n < R(s,t)$ this is zero, but the behavior becomes asymptotically interesting as n tends to infinity.

The (s,t)-Ramsey Multiplicity Problem

Denoting the density of s-cliques in a graph G by $c_s(G)$, determine

$$c_{s,t} = \lim_{n\to\infty} \min\{c_s(G) + c_t(\overline{G}) : v(G) = n\}.$$

Goodman [3] established that $c_{3,3} = 1/4$, **the same value as given by a random graph**. This motivated the following famous conjecture by Erdős when $s = t$.

Conjecture of Erdős (1962)

$c_{t,t} = 2^{1-\binom{t}{2}}$ for any integer t.

This was **refuted by Thomason [12]** for all $t > 3$. In particular for $t = 4$, he established that $c_{4,4} \leq 0.03029 < 0.03125 = 2^{1-6}$. It has also been shown that $c_{4,4} > 0.0296$. His constructions have received much attention over the years, but have proven difficult to improve upon.

Our Results

We obtain the new bounds

$$c_{4,4} < 0.03015$$

and

$$0.001524 < c_{5,5} < 0.001708.$$

We also study the off-diagonal case and prove that

$$c_{3,4} = 689 \cdot 3^{-8} \quad \text{and} \quad c_{3,5} = 24011 \cdot 3^{-12}.$$

Upper bounds are established through computer search heuristics and lower bounds using Flag Algebra SDP formulations, **resulting in a fully computer-assisted approach** [7].

Upper Bounds through Search Heuristics

Any finite sized graph can serve as an upper bound through its sequence of blow-ups. To drastically reduce the search space **we focused on constructing Cayley graphs**. Most previous constructions are in fact Cayley graphs. For a given group $\mathbf{G}$ of order n, let $\mathbf{s} \in \{\mathbf{0},\mathbf{1}\}^{\mathbf{N}}$ represent a symmetric generating set $S \subseteq \mathbf{G} \setminus \{0\}$ and $G_{\mathbf{s}}$ Cayley graph constructed this way. The optimization problem we are interested in can be described as follows:

The optimization problem for upper bounds

$$c_{s,t} \leq \min_{\text{group } \mathbf{G}} \min_{\mathbf{s}\in\{\mathbf{0},\mathbf{1}\}^{\mathbf{N}}} \frac{k!\, c_s(G_{\mathbf{s}}) \binom{n}{k}}{n^k} + \sum_{j=1}^{t} \frac{j!\, S(t,j)\, c_j(\overline{G_{\mathbf{s}}}) \binom{n}{j}}{n^t}.$$

Metaheuristics are a natural fit for this problem. We focused on two well-established probabilistic local search methods: **Simulated Annealing** [5] avoids getting stuck by also accepting slightly worse states according to some cooling criterion and **Tabu Search** [2] achieves the same by moving to the best neighbouring state that has not been visited recently.

The construction for our upper bound on $c_{4,4}$ was found in the group $C_3^{\times 2} \times C_2^{\times 5}$ and has 768 vertices and for $c_{5,5}$ it was found in $C_3 \times C_2^{\times 6}$ and has 192 vertices. The matching upper bounds for $c_{3,4}$ and $c_{3,5}$ are given by the Schläfli graph and its complement.

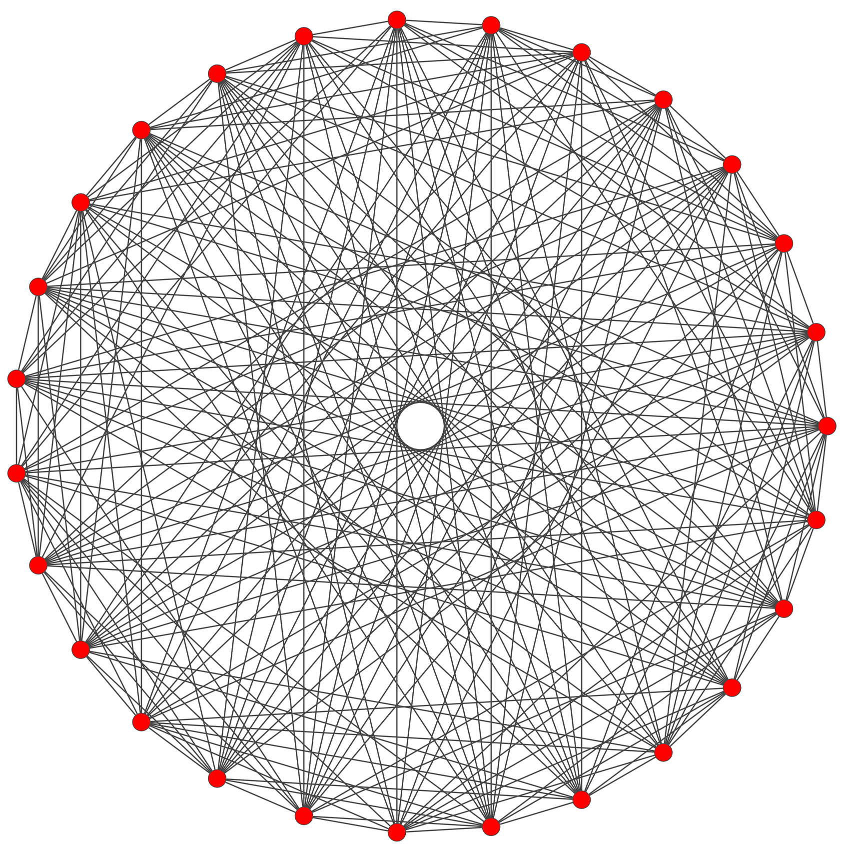

Figure 1. The Schläfli graph is a strongly regular graph on 27 vertices.

Lower Bounds through SDPs

Razborov [9] phrased this type of problem in the language of finite model theory, **allowing one to derive lower bounds through semidefinite programming**. For any integer $m \in \mathbb{N}$, we let $\mathcal{G}_m$ denote all graphs of order m and D_H a matrix representing particular pair densities of smaller graphs in $H \in \mathcal{G}_m$. We are interested in solving the following semidefinite problem:

The optimization problem for lower bounds

$$c_{s,t} \geq \max_{Q\succeq 0} \min_{H\in\mathcal{G}_m} c_s(H) + c_t(H) - \langle Q, D_H \rangle.$$

The size of the problem and the quality of the bound the SDP solver can produce is predominantly determined by m. Currently the largest realistically solvable appears to be $m = 8$, though our tight bounds for $c_{3,4}$ and $c_{3,5}$ are established already with $m = 6$. Beyond the asymptotics this approach can also give information about the **uniqueness of the constructions and stability** [8]. We show that anything that comes close to $c_{3,4}$ must be close to a symmetric blow-up of the Schläfli graph.

Restricting the Independence Number

One can **restrict the independence number** and ask for the asymptotic minimum density of independents sets of size s in that case, that is we want to determine

$$g_{s,t} = \lim_{n\to\infty} \min\{c_s(G) : v(G) = n, c_t(\overline{G}) = 0\}.$$

So far only $g_{3,s}$ and $g_{s,3}$ for $3 \leq s \leq 7$ are known. We prove that

$$g_{4,5} = 29 \cdot 13^{-3},$$

and also establish stability with respect to symmetric blow-ups of the unique $(3,5)$-Ramsey graph on 13 vertices.

Characterizing the Whole Region

Determining these parameters is **part of a more general questions where one is interested in characterizing the entire region** of points in $[0,1]^2$ than can be realized as densities of cliques and independent sets of a sequence of graphs. Define the region $\Omega_{s,t}$ as all $(x,y) \in [0,1]^2$ for which there exists a sequence of n-vertex graphs $(G_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} k_s(\overline{G_n}) \Big/ \binom{n}{s} = x \quad \text{and} \quad \lim_{n\to\infty} k_t(G_n) \Big/ \binom{n}{t} = y$$

We show that $\Omega_{s,t}$ **is compact, simply connected, and its upper and lower bounding curves are decreasing, continuous, and almost everywhere differentiable**. $\Omega_{2,t}$ was famously characterized by Razborov [10] for $t = 3$, Nikiforov [6] for $t = 4$, and Reiher [11] in general. $\Omega_{3,3}$ was characzerized by Huang et. al. [4]. We illustrate our findings on $\Omega_{3,4}$ in Figure 2. The blue line is the linear lower bound $y = c_{3,4} - x$, the blue dotted line is an additional linear lower bound for, the red dots represent optimal constructions, and the grey dots represent additional constructions. **Somewhat surprisingly, the constructions seems to get more complex into either direction starting from the Schläfli graph.**

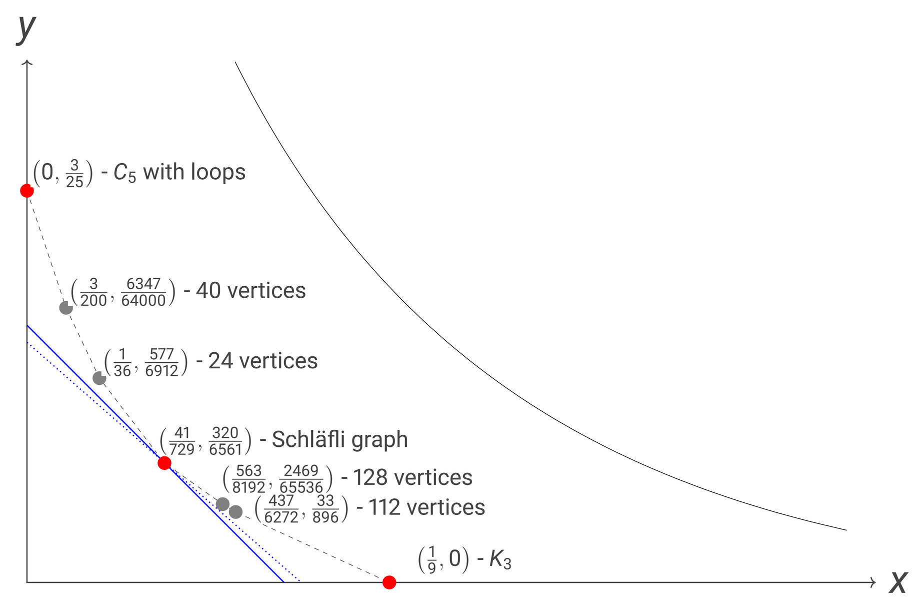

Figure 2. What is known about the region $\Omega_{3,4}$

Discussion and Outlook

The main goal of this work is to use a computer driven search to obtain constructions of graphs with small density of cliques of order s and density of independent sets of order t. **The most important question regarding the Ramsey multiplicity of K_4 is less its exact value, but rather whether it is given by the blow-up of a finite graph.** We believe that our answer to the $c_{3,4}$ problem provides some support to the possibility that the value of $c_{4,4}$ will also be determined by the blow-up sequence of a single graph.

Regarding future work that could build upon the presented tools, we believe there are three major points of interest:

- Using **different optimization methodologies** besides the mentioned search heuristics, the upper bounds derived from the optimization problems relating to $c_{4,4}$ and $c_{5,5}$ could be further improved.
- Using **different constructive approaches**, i.e., generalizing the notion of blow-ups or using other constructions besides Cayley graphs as the base, further improvements or even solutions to $c_{4,4}$ and $c_{5,5}$ could be obtained. It is quite likely that an immediate improvement can be gained from the found constructions using an iterative blow-up construction as done by Even-Zohar and Linial [1].
- There is a **large number of important but distinct problems in Extremal Combinatorics** besides the ones explicitly studied here, where the best current bounds are obtained by concrete and finitely describable constructions. It would be of great interest to see the methodologies applied here to the Ramsey Multiplicity problem and its variants also find application there.

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